Frontiers of Network Science
Fall 2024

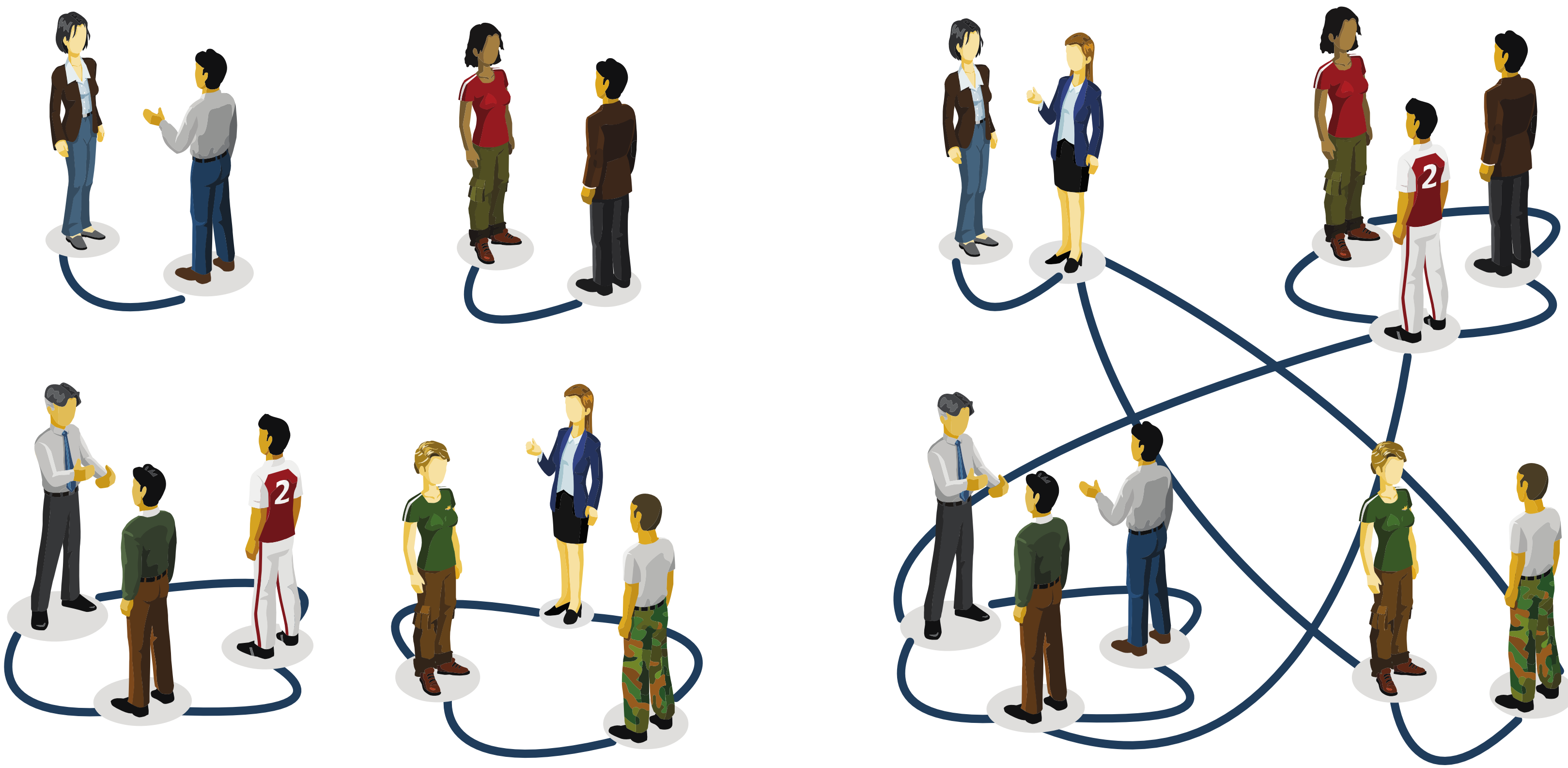
Class 4B: Random Networks
(Chapter 3 in Textbook)

Boleslaw Szymanski

based on slides by Albert-László Barabási & Roberta Sinatra

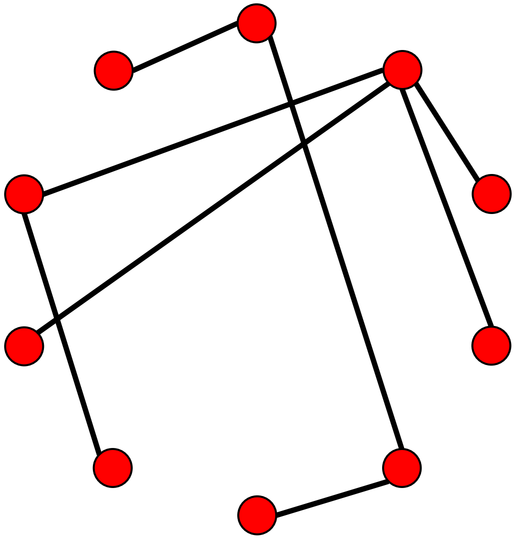
Introduction to Random Networks

RANDOM NETWORK MODEL

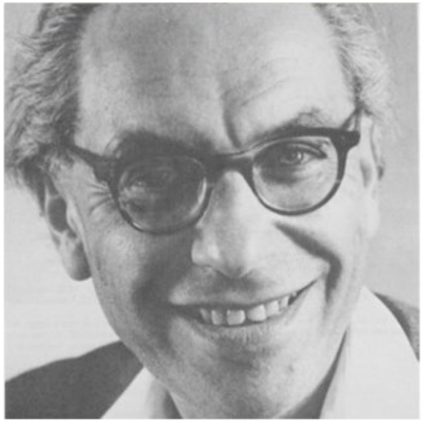


The random network model

RANDOM NETWORK MODEL



Pál Erdös
(1913-1996)



Alfréd Rényi
(1921-1970)

Erdös-Rényi model (1960)

Connect with probability p

p=1/6 N=10

<k> ~ 1.5

RANDOM NETWORK MODEL

Definition:

A **random graph** is a graph of N nodes where each pair of nodes is connected by probability **p**.

***G*(*N*, *L*) Model**

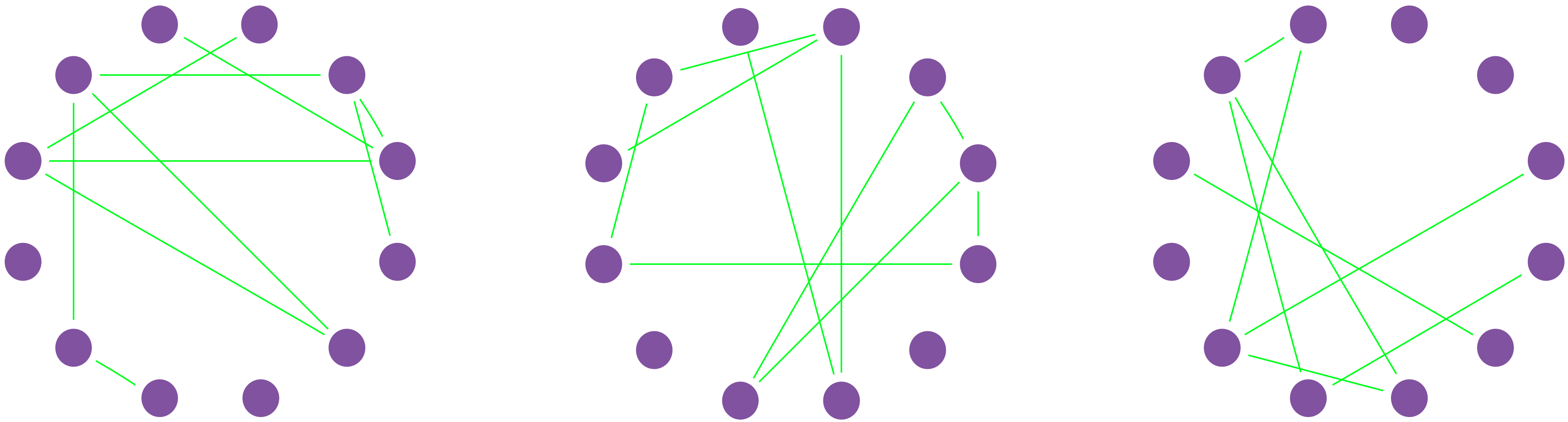
N labeled nodes are connected with *L* randomly placed links. Erdős and Rényi used this definition in their string of papers on random networks [2-9].

***G*(*N*, *p*) Model**

Each pair of N labeled nodes is connected with probability *p*, a model introduced by Gilbert [10].

RANDOM NETWORK MODEL

p=1/6
N=12



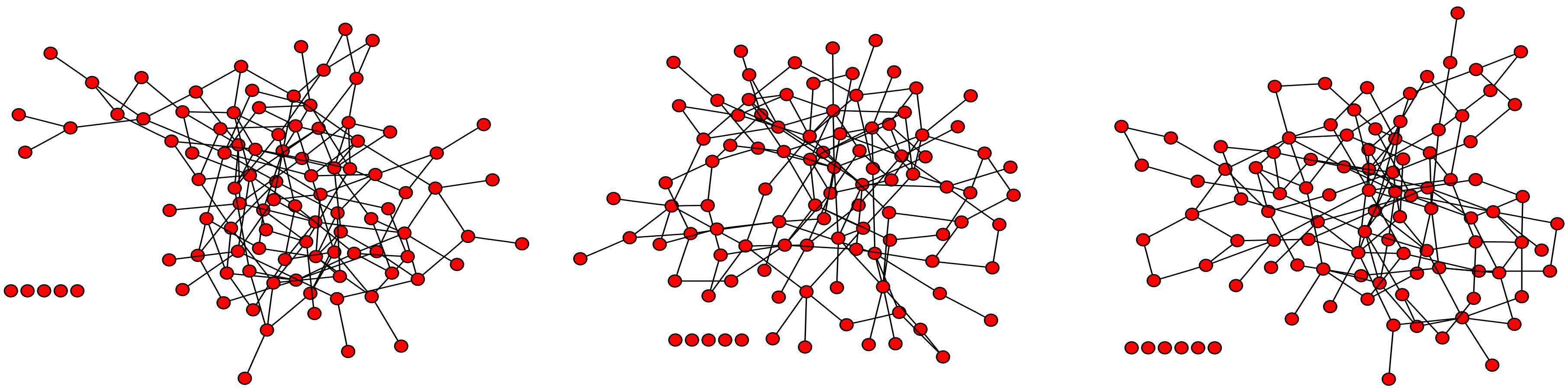
L=8

L=10

L=7

RANDOM NETWORK MODEL

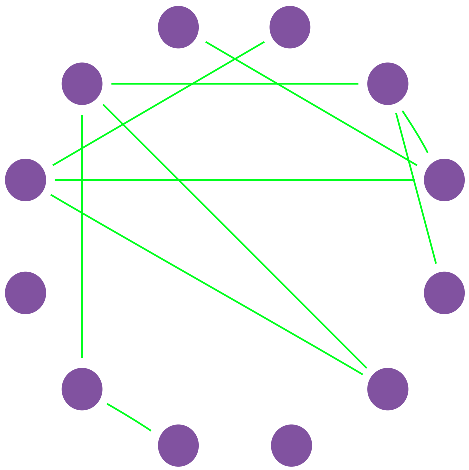
p=0.03
N=100



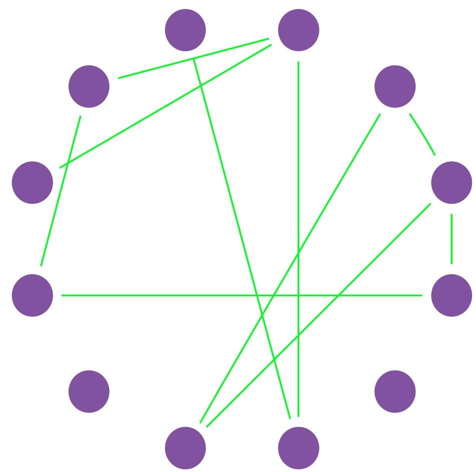
The number of links is variable

RANDOM NETWORK MODEL

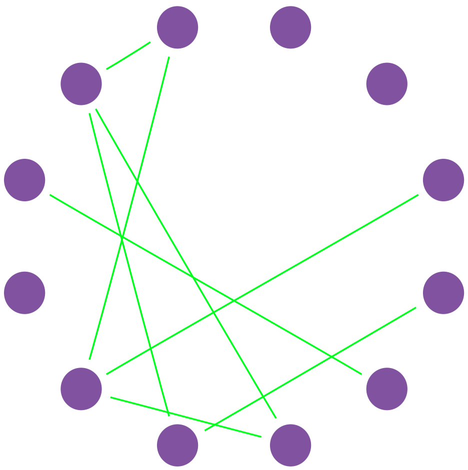
p=1/6
N=12



L=8



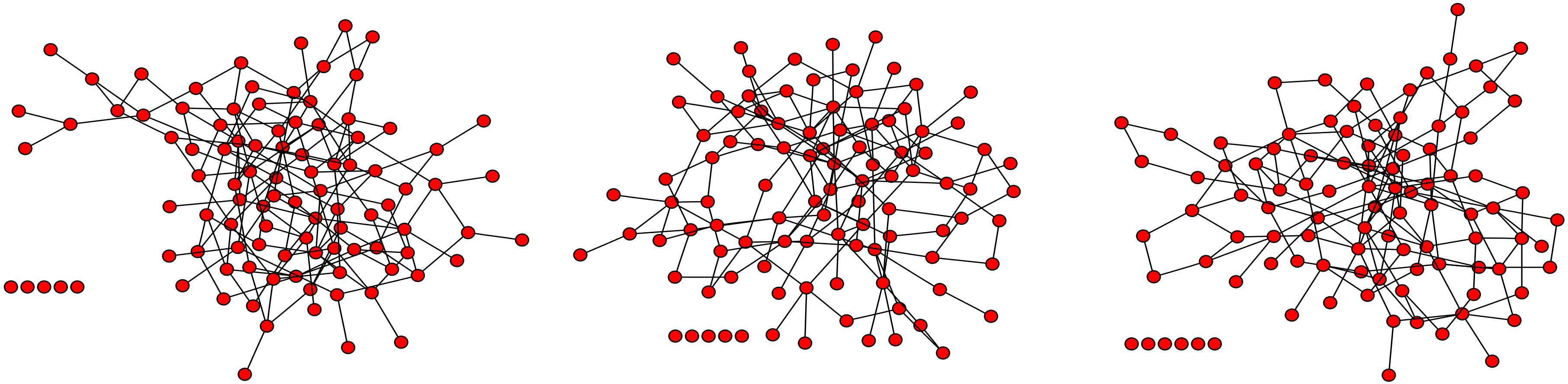
L=10



L=7

RANDOM NETWORK MODEL

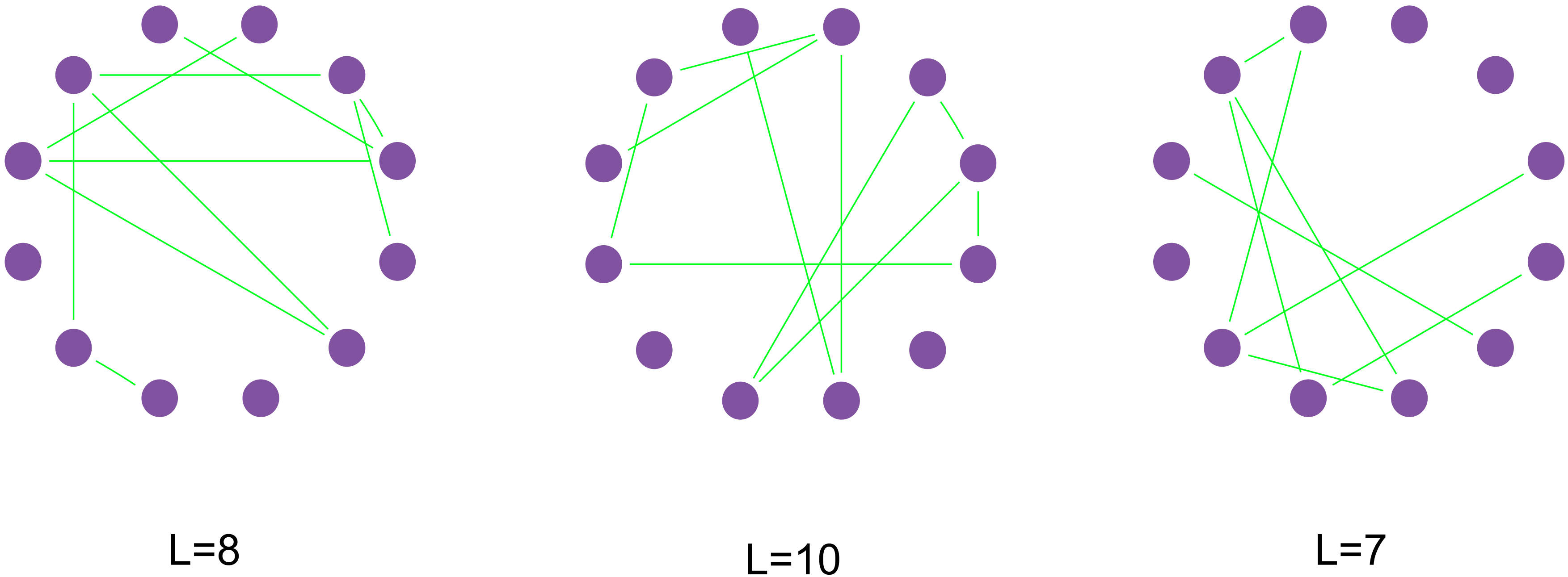
p=0.03
N=100



The number of links is variable

RANDOM NETWORK MODEL

p=1/6
N=12



Number of links in a random network

P(L): the probability to have exactly *L* links in a network of *N* nodes and probability *p*:

The maximum number of links in a network of N nodes.

$$P(L) = \binom{\binom{N}{2}}{L} p^L (1-p)^{\frac{N(N-1)}{2}-L}$$

Number of different ways we can choose L links among all potential links.

Binomial distribution...

MATH TUTORIAL | Binomial Distribution: The bottom line

$$P(x) = \binom{N}{x} p^x (1-p)^{N-x}$$

$$< x >= Np$$

$$< x^2 >= p(1-p)N + p^2 N^2$$

$$\sigma_x = (< k^2 > - < k >^2)^{1/2} = [p(1-p)N]^{1/2}$$

http://keral2008.blogspot.com/2008/10/derivation-of-mean-and-variance-of.html

RANDOM NETWORK MODEL

P(L): the probability to have a network of exactly *L* links

$$P(L) = \binom{\binom{N}{2}}{L} p^L (1-p)^{\frac{N(N-1)}{2}-L}$$

- The average number of links <*L*> in a random graph

$$< L >= \sum_{L=0}^{\frac{N(N-1)}{2}} LP(L) = p\frac{N(N-1)}{2}$$

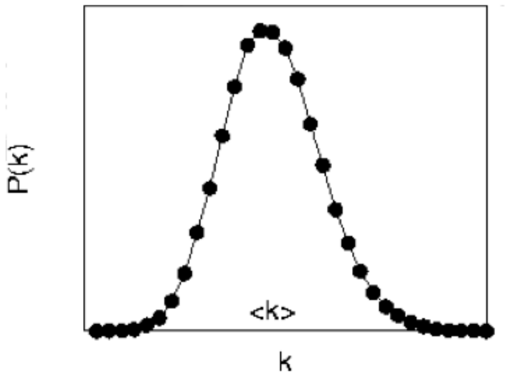
$$< k >= 2L/N = p(N-1)$$

- The standard deviation

$$\sigma^2 = p(1-p)\frac{N(N-1)}{2}$$

Degree distribution

DEGREE DISTRIBUTION OF A RANDOM GRAPH



$$P(k) = \binom{N-1}{k} p^k (1-p)^{(N-1)-k}$$

Select k nodes from N-1

probability of having *k* edges

probability of missing N-1-k edges

$$<k> = p(N-1) \qquad \sigma_k^2 = p(1-p)(N-1)$$

$$\frac{\sigma_k}{<k>} = \left[\frac{1-p}{p}\frac{1}{(N-1)}\right]^{1/2} \approx \frac{1}{(N-1)^{1/2}}$$

As the network size increases, the distribution becomes increasingly narrow—we are increasingly confident that the degree of a node is in the vicinity of <k>.

DEGREE DISTRIBUTION OF A RANDOM GRAPH

$$P(k) = \binom{N-1}{k} p^k (1-p)^{(N-1)-k}$$

$$<k> = p(N-1)$$

$$p = \frac{<k>}{(N-1)}$$

For large *N* and small *k*, we can use the following approximations:

$$\binom{N-1}{k} = \frac{(N-1)!}{k!(N-1-k)!} = \frac{(N-1)(N-1-1)(N-1-2)\ldots(N-1-k+1)(N-1-k)!}{k!(N-1-k)!} = \frac{(N-1)^k}{k!}$$

$$\ln[(1-p)^{(N-1)-k}] = (N-1-k)\ln(1-\frac{<k>}{N-1}) = -(N-1-k)\frac{<k>}{N-1} = -<k>(1-\frac{k}{N-1}) \cong -<k>$$

$$(1-p)^{(N-1)-k} = e^{-<k>}$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots \quad \text{for} \quad |x| \le 1$$

$$P(k) = \binom{N-1}{k} p^k (1-p)^{(N-1)-k} = \frac{(N-1)^k}{k!} p^k e^{-<k>} = \frac{(N-1)^k}{k!} \left(\frac{<k>}{N-1}\right)^k e^{-<k>} = e^{-<k>} \frac{<k>^k}{k!}$$

POISSON DEGREE DISTRIBUTION

$$P(k) = \binom{N-1}{k} p^k (1-p)^{(N-1)-k}$$

$$<k> = p(N-1)$$

$$p = \frac{<k>}{(N-1)}$$

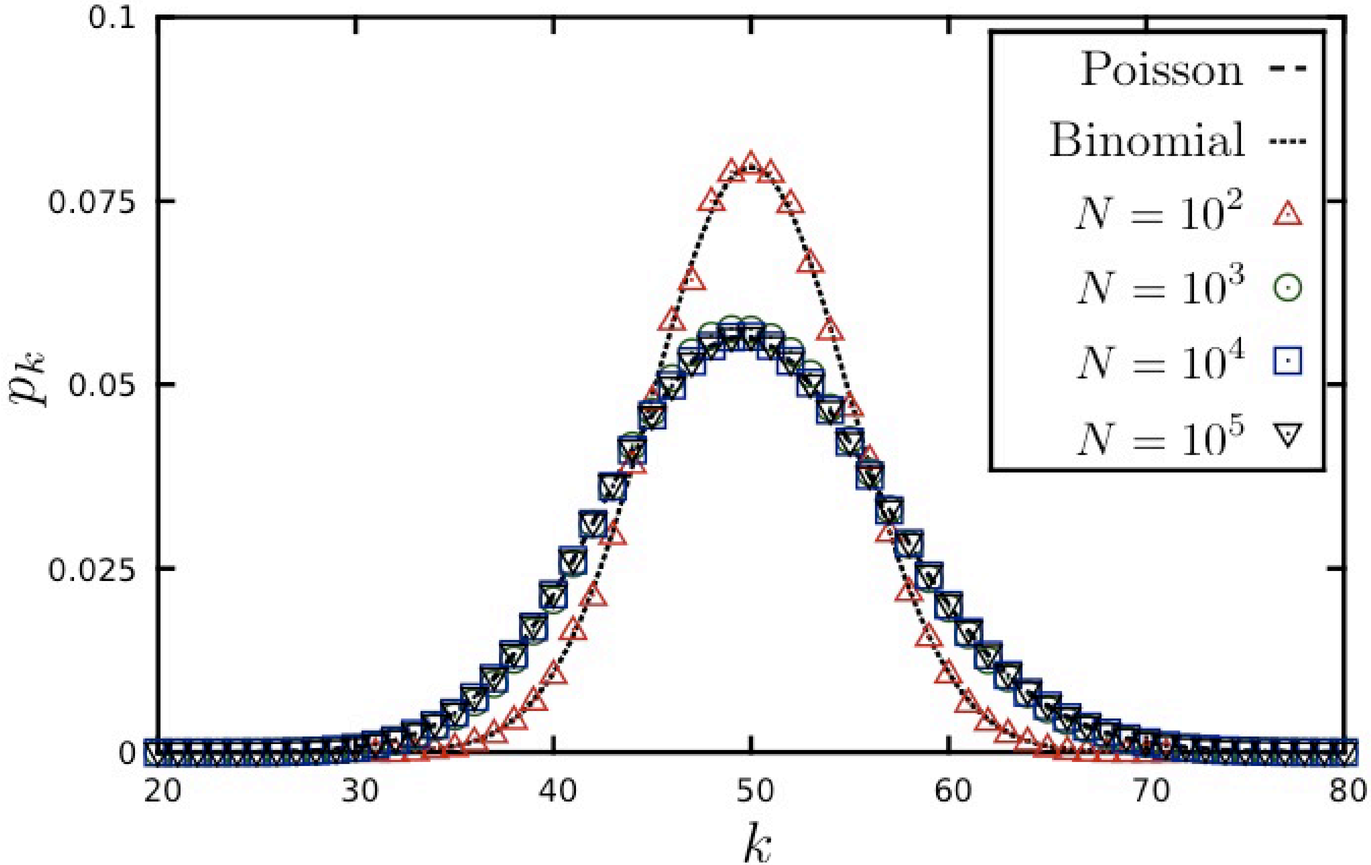
For large *N* and small *k*, we arrive at the Poisson distribution:

$$P(k) = e^{-<k>} \frac{<k>^k}{k!}$$

DEGREE DISTRIBUTION OF A RANDOM GRAPH

<k>=50

$$P(k) = e^{-<k>} \frac{<k>^k}{k!}$$



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